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APPROXIMATIONS IN L^p AND TCHEBYCHEFF APPROXIMATIONS

by

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APPROXIMATIONS IN L^p AND TCHEBYCHEFF APPROXIMATIONS

One of the difficulties of the analysis of Tchebycheff approximations for functions of several variables lies in the fact that the best approximation is not always unique [Ref. 1]. Consequently it can be useful to define a reasonable particular unique Tchebycheff approximation. In Reference 1, J. R. Rice defines a unique "strict approximation," but only for functions defined on a finite set. In this report, we show that the strict approximation is the limit of the best approximation in L^p when $p \to \infty$. Unfortunately, this relation does not help to solve the initial problem, since we shall show an example of a continuous function of two variables defined on a rectangle such that the best linear approximation in L^p does not converge for $p \to \infty$.

Approximations on a finite set.

Let $F(x)$, $g_1(x)$, ... $g_n(x)$ be functions defined on the finite set $E = \{x_1, \ldots x_m\}$. We suppose that the matrix $||f_i(x_j)||$ has the rank n. Using the same notations as in Reference 1, for any set of parameters a_1, ... a_n,

$$L(A,x) = \sum_{i=1}^{n} a_i g_i(x) = [A,G(x)]$$

represents an *approximation*; A denotes the vector of components a_1, ... a_n; $G(x)$ is the vector of components $g_1(x)$, ... $g_n(x)$; $[A, G(x)]$ is the scalar product of A and $G(x)$. $r(A,x) = L(A,x) - f(x)$ is the *error of the approximation*.

Tchebycheff approximation. $L(A,x)$ is a best Tchebycheff approximation, if the parameters a_1, ... a_n minimize the expression

$$\max_{i=1,\ldots m} |r(A,x_i)|$$

Notations. The *deviation* ρ of $F(x)$ relatively to $G(x)$ on a set $S \subset E$ is given by

$$\rho = \inf_A \max_{x \in S} |r(A,x)| .$$

Let W be the set of all best Tchebycheff approximations of $F(x)$ relatively to $G(x)$ on S. The characteristic set R of $F(x)$ relatively to $G(x)$ on S is the set of points $x \in S$ such that $|r(A,x)| = \rho$ for all approximations $L(A,x) \in W$.

Remark. In the terminology of Reference 1, R is the set of all "critical point sets" and in the terminology of Reference 2, R is the union of all "cadres maxima."

Lemma 1. a) W is non-void;

b) R is non-void;

c) if for a given approximation $L(A,x)$, there exists a point $\bar{x} \in R$ such that $r(A,\bar{x}) \neq \rho$, then $\max_{x \in R} |r(A,x)| > \rho$.

For the proof of lemma 1, see Reference 2, theorem 7.

Strict approximation. The strict approximation $L(A^*,x)$ is defined by the following construction: let ρ_1, W_1, R_1 be respectively the deviation, the set of best approximations, the characteristic set of $F(x)$ relatively to to $G(x)$ on $E = \left\{x_1, \dots x_m\right\}$. Let s_1 be the rank of the set of vectors $\left\{G(x) \middle| x \in R_1\right\}$; (it may happen that $s_1 = 0$). By the convexity of W_1, if $x \in R_1$, then $r(A,x) = \epsilon(x)\rho_1$ for all $L(A,x) \in W_1$ with $\epsilon(x) = \pm 1$ independent of $L(A,x)$. Consequently, W_1 is a subset of the solutions of the system

$$L(A,x) - F(x) = \epsilon(x)\rho_1, \; x \in R_1 \qquad (1)$$

Now we consider the best approximations of $F(x)$ relatively to $G(x)$ on $E - R_1$, but subject to the condition (1). Since (1) has the rank s_1, we can eliminate s_1 parameters, let us say $a_{m-s_1+1}, \dots a_m$; the approximations take the form

$$L^{(2)}(A,x) = \sum_{i=1}^{m-s_1+1} a_i g_i^{(2)}(x)$$

and the new problem is to approximate $F(x)$ by $L^{(2)}(A,x)$ on $E - R_1$. Let $G^{(2)}(x) = (g_1^{(2)}(x), \dots g_{m-s_1+1}^{(2)}(x))$ and ρ_2, W_2, R_2 be respectively the deviation, the set of best approximations, the characteristic set of $F(x)$ relatively to

$G^{(2)}(x)$ on $E - R_1$; s_2 is the rank of the set of vectors $\left\{G^{(2)}(x) \middle| x \in R_2\right\}$. It is clear that $\rho_2 \leq \rho_1$ and $W_2 \subset W_1$. We show that $\rho_2 < \rho_1$; suppose $\rho_2 = \rho_1$; then any approximation of W_1 belongs also to W_2 and $W_1 \subset W_2$; by a previous relation $W_1 = W_2$ and $R_2 \subset R_1$ which is a contradiction by lemma 1b. W_2 is a subset of the solutions of the system

$$L^{(2)}(A,x) - F(x) = \epsilon(x)\rho_2, \quad x \in R_2 \qquad (2)$$

where $\epsilon(x) = \pm 1$ is such that (2) is verified for an approximation $L(A,x) \in W_2$.

We consider the best approximations of $F(x)$ relatively to $G(x)$ on $E - R_1 - R_2$, but subject to the conditions (1) and (2). With the help of (2), we can eliminate s_2 new parameters, let us say $a_{n-s_1-s_2+1}$, ... a_{n-s_1} and the new approximations are of the form

$$L^{(3)}(A,x) = \sum_{i=1}^{n-s_1-s_2} a_i g_i^{(3)}(x);$$

the new problem is to approximate $F(x)$ by $L^{(3)}(A,x)$ on $E - R_1 - R_2$.

Clearly, we can continue inductively this process and get the sequences $\rho_1 > \rho_2, \ldots > \rho_k$, $L^{(1)}(A,x) = L(A,x)$, $L^{(2)}(A,x)$, ... $L^{(k)}(A,x)$, $R_1, R_2, \ldots R_k$, until $s_1 + s_2 + \ldots + s_k = n$, which will happen once since the rank of the matrix $||g_i(x_k)||$ is n. The set of equations (1), (2) ... will define a unique best approximation called "strict approximation."

Remark. The construction of the strict approximation in the original paper Reference 1, is slightly different but entirely equivalent.

Approximation in L^p. $L(A,x)$ is a best approximation in L^p of $F(x)$ on a set $S \subset E$ relatively to $G(x)$ if the parameters a_1, ... a_n minimize the expression

$$\sum_{x \in S} |r(A,x)|^p .$$

Lemma 2. a) if the rank of the vectors $\left\{G(x) \middle| x\in S\right\}$ is n, there exists a unique best approximation in L^p for $p > 1$;

b) If the rank of the vectors $\left\{G(x) \middle| x\in S\right\}$ is $t < n$, the best approximations in L^p, $p > 1$, form a hyperplan of dimension $n - t$; if $L(A,x)$ and $L(A',x)$ are two best approximations, then $r(A,x) = r(A',x)$ for $x\in S$.

For the proof of lemma 2a, see Reference 3, Chapter I. Lemma 2b is a consequence of lemma 2a; since the rank of the matrix of elements $g_i(x)$ for $x\in S$, $i = 1, \ldots n$, is t, there exist t functions, let us say $g_1(x), \ldots g_t(x)$, such that the rank of the same matrix without the elements corresponding to $i = t + 1, \ldots n$ is also t; let $L(\bar{A},x)$ be a best approximation; there exists a unique approximation of the form $L'(A',x) = \sum_{i=1}^{t} a'_i g_i(x)$ such that $L(\bar{A},x) = L'(A',x)$ for $x\in S$; but $L'(A',x)$ must be the unique best approximation of $F(x)$ by $g_1(x), \ldots g_t(x)$ on S; consequently $L(A,x) = L'(A',x)$ for $x\in S$ if and only if $L(A,x)$ is a best approximation of $F(x)$ by $g_1(x), \ldots g_n(x)$ on S; the lemma 2b follows immediately.

Remark. Let $1 < p_1 < p_2 < p_3 \ldots$ be a sequence growing to infinity and $L(A_j,x)$ be a best approximation in L^{p_j} of $F(x)$ on $S \subset E$ relatively to $G(x)$; it is rather obvious that the errors $r(A_j,x)$ for $x\in S$ are bounded; consequently, there exists a subsequence p_{i_j} such that $\lim_{j\to\infty} r(A_{i_j},x)$ exists for all $x\in S$.

Theorem. If $L(A^*,x)$ and $L(A_p,x)$ denote respectively the strict approximation and the best approximation in L^p of $F(x)$ on E relatively to $G(x)$, then

$$L(A^*,x) = \lim_{p\to\infty} L(A_p,x) .$$

Proof. Let $R_1, \ldots R_k$ be the characteristic sets defined in the construction of the strict approximation. We prove by induction that for $i = 1,2, \ldots k$

$$\lim_{p\to\infty} r(A_p,x) = r(A^*,x) , \quad x\in R_i ;$$

the theorem follows immediately since the rank of the vectors $\left\{G(x) \middle| x \in \cup R_i\right\}$ is n.

Let $K_p^{(i)} = \left\{L(A_p^{(i)}, x)\right\}$ be the set of the best approximations of F(x) in L^p on $C_i = \bigcup_{j=1}^{i} R_j$ relatively to G(x). In the i-cycle of induction, we shall also prove that

$$\lim_{p\to\infty} r(A_p^{(j)},x) = r(A^*,x) \ , \ x \in R_i \quad \text{for} \quad j = i, i+1, \ldots k.$$

First step: suppose that for an element $x' \in R_1$

$$\lim_{p\to\infty} r(A_p,x') \neq r(A^*,x'), \qquad \text{(or the limit does not exist);}$$

by lemma 1c and the remark of lemma 2, there exists an element $\bar{x} \in R_1$, a number z and a sequence $1 < p_1 < p_2 \ldots$ growing to infinity such that

$$r(A_{p_j},\bar{x}) > z > \rho_1 \quad \text{for} \quad j = 1,2, \ldots$$

we have the relations

$$\sum_{x\in E} |r(A^*,x)|^{p_j} \leq m\rho_1^{\,p_j} \quad \text{and} \quad \sum_{x\in E} |r(A_{p_j},x)|^{p_j} > z^{p_j} \ ,$$

where m is the number of elements of E. There exist N such that for $j > N$ $m\rho_1^{\,p_j} < z^{p_j}$; that means that $L(A^*,x)$ is a better approximation in L^{p_j} than $L(A_{p_j},x)$ which is a contradiction. The same argument can be applied for $L(A_p^{(j)},x)$ for $j = 1,2,\ldots k$ by replacing in the above equations A_p by $A_p^{(j)}$ and E by C_j. We have the results

$$\lim_{p\to\infty} r(A_p,x) = r(A^*,x) \ , \quad x \in R_1 = C_1 \ ;$$

$$\lim_{p\to\infty} r(A_p^{(j)},x) = r(A^*,x) \ , \quad x \in R_1 = C_1 \ .$$

Second step: suppose we have already proved that

$$\lim_{p\to\infty} r(A_p,x) = r(A^*,x) , \quad x\in C_{i-1} ; \tag{3}$$

$$\lim_{p\to\infty} r(A_p^{(j)},x) = r(A^*,x) , \quad x\in C_{i-1}, \quad \text{for} \quad j = i-1,i,\ldots k, \tag{4}$$

and suppose there exists an element $x'\in R_i$ for which

$$\lim_{p\to\infty} r(A_p,x') \neq r(A^*,x'), \quad \text{(or the limit does not exist);}$$

by lemma 1c, equation 3, and the remark of lemma 2, there exists a point $\bar{x}\in R_i$, a number $z > \rho_i$ and a sequence $1 < p_1 < p_2 \ldots$ growing to infinity such that

$$r(A_{p_j},\bar{x}) > z > \rho_i \quad \text{for} \quad j = 1,2\ldots \tag{5}$$

Let us define the distance $d(L(A,x), L(A',x))$ of two approximations as the euclidian distance between the vectors A and A'; by lemma 2b, there exists a unique approximation $L(\bar{A}_p^{(i-1)}, x)\in K_p^{(i-1)}$ which is the closest to $L(A^*,x)$ among all elements of $K_p^{(i-1)}$; by 3

$$\lim_{p\to\infty} L(\bar{A}_p^{(i-1)},x) = L(A^*,x);$$

there exists N and w such that for $j > N$

$$r(\bar{A}_{p_j}^{(i-1)},x) < w < z \quad \text{for} \quad x\in E - C_{i-1};$$

using this relation and the fact that $L(\bar{A}_{p_j}^{(i-1)},x)\in K_{p_j}^{(i-1)}$, we have

$$\sum_{x\in E} |r(\bar{A}^{(i-1)}_{p_j},x)|^{p_j} = \sum_{x\in C_{i-1}} |r(\bar{A}^{(i-1)}_{p_j},x)|^{p_j} + \sum_{x\in E-C_{i-1}} |r(\bar{A}^{(i-1)}_{p_j},x)|^{p_j}$$

$$\leq \sum_{x\in C_{i-1}} |r(A_{p_j},x)|^{p_j} + mw^{p_j}; \qquad (6)$$

by (5):

$$\sum_{x\in E} |r(A_{p_j},x)|^{p_j} \geq \sum_{x\in C_{i-1}} |r(A_{p_j},x)|^{p_j} + z^{p_j}; \qquad (7)$$

there exists $M \geq N$ such that

$$mw^{p_j} < z^{p_j} \quad \text{for} \quad j > M \qquad (8)$$

by (6), (7), (8):

$$\sum_{x\in E} |r(\bar{A}^{(i-1)}_{p_j},x|^{p_j} < \sum_{x\in E} |r(A_{p_j},x)|^{p_j} \quad \text{for} \quad j > M,$$

which contradicts the fact that $L(A_{p_j},x)$ is the best approximation in L^{p_j} of $F(x)$ on E. The same argument can be applied for $L(A^{(j)}_p,x)$, $j = i,i+1,\ldots k$ by replacing A_p by $A^{(j)}_p$, E by C_j. We have the desired results:

$$\lim_{p\to\infty} r(A_p,x) = r(A^*,x), \quad x\in C_i ;$$

$$\lim_{p\to\infty} r(A^{(j)}_p,x) = r(A^*,x), \quad x\in C_i, \quad \text{for} \quad j = i,i+1,\ldots k.$$

A disappointing example.

We shall define a continuous function $F(x,y)$ on the rectangle $-2 \leq x \leq 2$, $-4 \leq y \leq 4$ such that the coefficients of the best approximation in L^p $a_p x + b_p y + c_p$ do not converge when p grows to infinity; (an example of a function of one variable with approximations of the form ax would be slightly simpler).

The function $D(y)$ is clearly defined by the Figure 1 for $-4 \leq y \leq 4$. The function $G(x)$ which will be constructed later has the following properties (Figure 3):

1) $G(x)$ is defined for $-2 \leq x \leq 2$;
2) $G(x)$ is increasing for $-2 \leq x \leq 0$, decreasing for $0 \leq x \leq 2$;
3) $G(-2) = G(2) = 2$; $G(0) = 6$;
4) $G(x) \leq 6 - 2\,|x|$;

then

$$F(x,y) = \begin{cases} \min\,(D(y),\ G(x)) & \text{for} \quad -4 \leq y \leq -2 \quad \text{and} \quad 2 \leq y \leq 4; \\ \max\,(D(y),\ -G(-x)) & \text{for} \quad -2 \leq y \leq 2; \end{cases}$$

in other words Figure 1 and Figure 2 represent the orthogonal projection of the surface $z = F(x,y)$ on the plans (y,z) and (x,z).

Let $z_p = a_p x + b_p y + c_p$ be the best approximation of $F(x,y)$ in L^p. The plan $z'_p = a_p x - b_p y + c_p$, image of z_p with respect to the plan $y = 0$, is also a best approximation; since the best approximation is unique (cf Reference 3), $z_p = a_p x + c_p$; furthermore, the approximation $z''_p = a_p x - c_p$, image of z'_p with respect to the line $x = 0$, $z = 0$, is also a best approximation and therefore the best approximation is of the form $z_p = a_p x$. The equation

$$\int_{-4}^{4} dy \int_{-2}^{2} |F(x,y) - ax|^p \, dx = \int_{-2}^{2} |G(x) - ax|^p \left(8 - \frac{2}{3}\,G(x)\right) dx = I(-2,2,a,p,G(x))$$

reduces the problem to a single variable.

The function $G(x)$ is defined by recurrence; at the k-step, the quantities u'_k, u_k, l'_k, l_k, v'_k, v_k, z'_k, z_k, p'_k, p_k will be determined; they will satisfy the relations

$$u_o = -2; \; v_o = 2; \; z_o = 2;$$

$$u_{k-1} < l'_k < u'_k < u_k; \qquad (10)$$

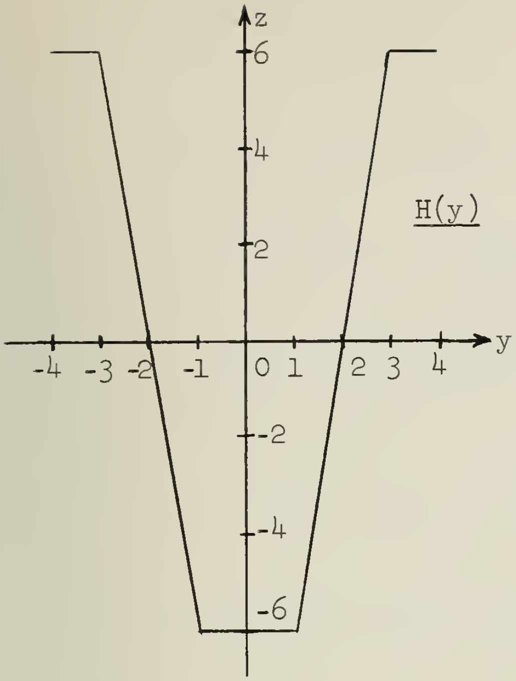

Figure 1

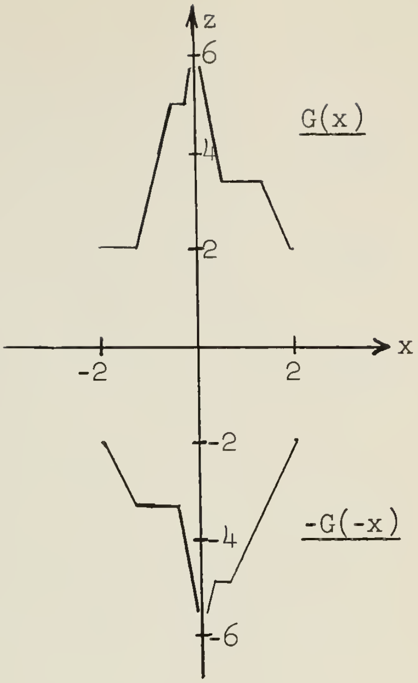

Figure 2

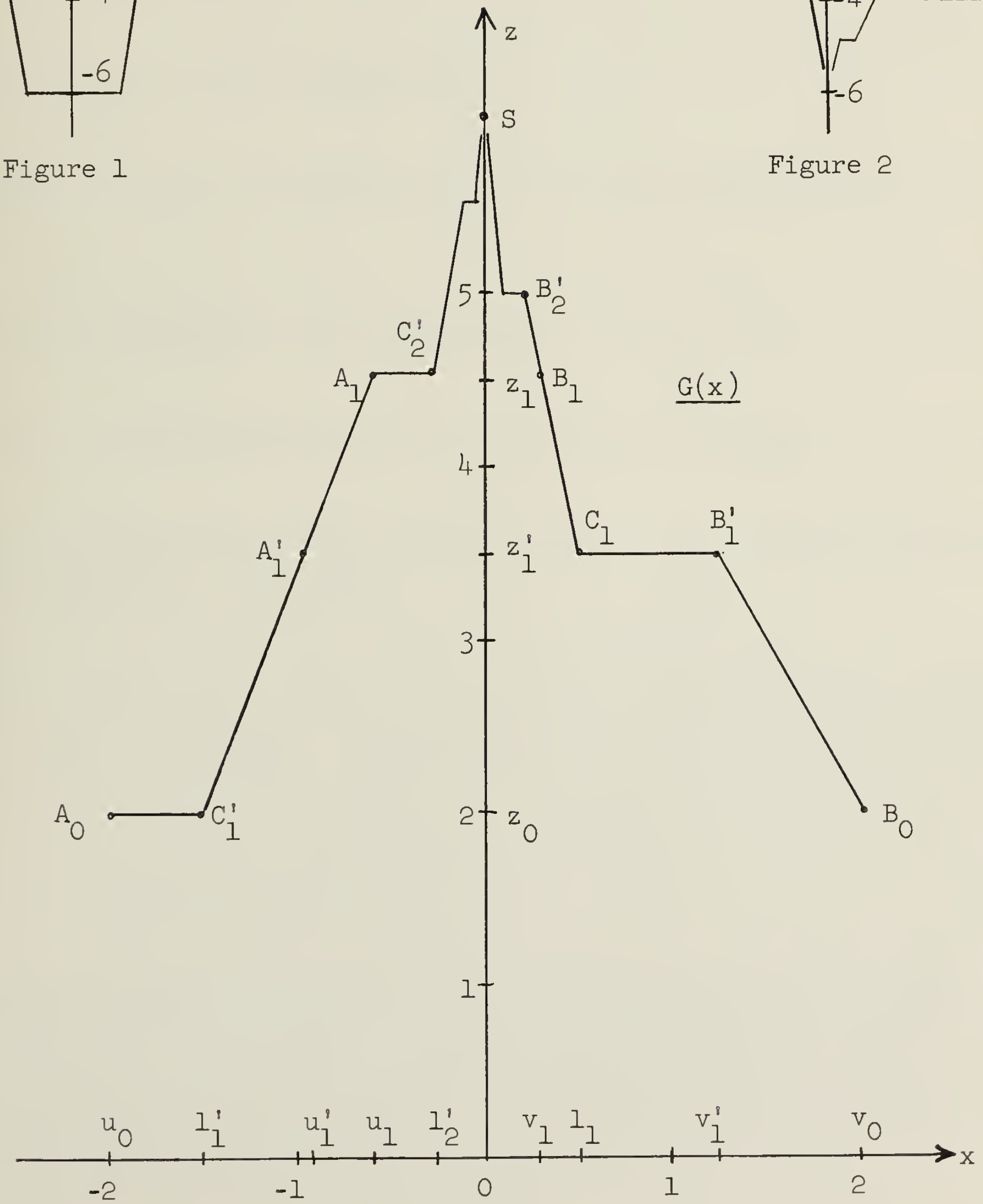

Figure 3

$$v_{k-1} > v'_k > l_k > v_k;$$

$$z_{k-1} < z'_k < z_k; \quad \frac{6(k-1)}{k} < z_k < 6; \tag{11}$$

$$p_{k-1} < p'_k < p_k; \quad p_k > k; \tag{12}$$

$$\lim_{k\to\infty} u_k = \lim_{k\to\infty} v_k = 0; \tag{13}$$

$$z'_k = G(v'_k) = G(u'_k) = G(l_k);$$

$$z_k = G(v_k) = G(u_k) = G(l'_{k+1});$$

at the k-step, $G(x)$ will be defined for $u_{k-1} \leq x \leq u_k$ and $v_k \leq x \leq v_{k-1}$ such that

$$I(-2,2,1,p'_k,G(x)) \leq I(-2,2,a,p'_k,G(x)) \quad \text{for} \quad a < 0.5; \tag{14}$$

$$I(-2,2,-1,p_k,G(x)) \leq I(-2,2,a,p_k,G(x)) \quad \text{for} \quad a > -0.5; \tag{15}$$

the equations (14) and (15) prove that $a_{p'_k} > 0.5$, $a_{p_k} < -0.5$ and consequently the asserted phenomenal of divergence.

Construction. Suppose $G(x)$ constructed for $-2 \leq x \leq u_{k-1}$ and $v_{k-1} \leq x \leq 2$; let $A_{k-1} = (u_{k-1}, z_{k-1})$, $B_{k-1} = (v_{k-1}, z_{k-1})$ and $S = (0,6)$, (Figure 3.) We join B_{k-1} to S by a straight line; let $-\alpha$ be its slope and $\beta > \alpha$ such that

$$(\alpha + 0.5)\left(\frac{1}{\beta - 1} + \frac{1}{\alpha + 1}\right) < 1 . \tag{16}$$

Let $C'_k = (l'_k, z_{k-1})$ be the intersection of the straight line passing by S of slope β with the horizontal line $z = z_{k-1}$; the function $H(x)$ is equal to $G(x)$ for $-2 \leq x \leq u_{k-1}$, $v_{k-1} \leq x \leq 2$ and is defined by the polygon $A_{k-1}C'_kSB_{k-1}$ for $u_{k-1} \leq x \leq v_{k-1}$; p'_k is chosen such that

$$I(-2,2,1,p'_k,H(x)) < I(0,v_{k-1},0.5,p'_k,H(x)); \tag{17}$$

let for $z_{k-1} < c < 6$ the function $M(x)$ be defined by the relations

$$M(x) = \begin{cases} H(x) & \text{for} \quad H(x) \le c; \\ c & \text{for} \quad H(x) \ge c; \end{cases}$$

by (17) it is clear that there exists a value $\bar{c}$ for which

$$I(-2,2,1,p'_k,H(x)) \le I(0,v_{k-1},0.5,p'_k,M(x)); \tag{18}$$

then $z'_k \equiv \bar{c}$; $A'_k = (u'_k, z'_k)$ and $B'_k = (v'_k, z'_k)$ are the intersections of $z = H(x)$ with the horizontal line $z = z'_k$; finally $G(x)$ is defined as $H(x)$ for $z \le z'_k$, i.e., for $-2 \le x \le u'_k$ and $v'_k \le x \le 2$. We shall prove later that the choice of p'_k according to (17) is possible and that (14) follows from (17) and (18).

The construction of the second part of the step of induction is analogous to the first one and will be described briefly only. We join A'_k to S by a straight line; let δ be its slope and $\gamma > \delta$ such that

$$(\delta + 0.5)\left(\frac{1}{\gamma - 1} + \frac{1}{\delta + 1}\right) < 1;$$

let C_k be the intersection of the straight line passing by S of slope $-\gamma$ with the horizontal $z = z'_k$; p_k and z_k must satisfy equations similar to (17) and (18) in order that (15) holds (furthermore in order that $\lim\limits_{k\to\infty} p_k = \infty$ and $\lim\limits_{k\to\infty} z_k = 6$, (11) and (12) must be verified). Let $L(x)$ be equal to $G(x)$ for $-2 \le x \le u'_k$ and $v'_k \le x \le 2$ and be defined by the polygon $A'_k S C_k B'_k$ otherwise. The new function $G(x)$ is now defined as $L(x)$ for $L(x) \le z_k$; $A_k = (u_k, z_k)$ and $B_k = (v_k, z_k)$ are the intersections of $z = L(x)$ with the horizontal $z = z_k$.

The proof that for p'_k large enough (17) holds is a little tedious and will be given only briefly; we obtain the following results after some computations:

$$I(-2,2,1,p,H(x)) = I(-2,l'_k,1,p,H(x)) + I(l'_k,0,1,p,H(x))$$

$$+ I(0,v_{k-1},1,p,H(x)) + I(v_{k-1},2,1,p,H(x));$$

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$$I(-2,l_k',1,p,H(x)) + I(v_{k-1},2,1,p,H(x)) \leq 32w^p;$$

$$I(l_k',0,1,p,H(x)) \leq \frac{4 \cdot 6^{p+1}}{(p+1)(\beta-1)} \left(1 + \frac{K_1}{p+2}\right);$$

$$I(0,v_{k-1},1,p,H(x)) \leq \frac{4 \cdot 6^{p+1}}{(p+1)(\alpha+1)} \left(1 + \frac{K_2}{p+2}\right);$$

$$I(0,v_{k-1},0.5,p,H(x)) \geq \frac{4 \cdot 6^{p+1}}{(p+1)(\alpha+0.5)} \left(1 - K_3 \left(\frac{w}{6}\right)^p - \frac{K_4}{p+2} \left(\frac{w}{6}\right)^p\right)$$

where $w = 3 + 0.5\,z_k < 6$ (by equation (9)) and K_1, K_2, K_3, K_4 are independent of p; combining these last relations, we have

$$I(0,v_{k-1},0.5,p,H(x)) - I(-2,2,1,p,H(x))$$

$$\geq \frac{4 \cdot 6^p}{(p+1)(\alpha+0.5)} \left\{1 - (\alpha+0.5)\left(\frac{1}{\beta-1} + \frac{1}{\alpha+1}\right) - \frac{1}{p+2}\left(Q_1 + K_4\left(\frac{w}{6}\right)^p\right) - K_3\left(\frac{w}{6}\right)^p - Q_2(p+1)\left(\frac{w}{6}\right)^p\right\}$$

where Q_1 and Q_2 are independent of p; by (16), for p large enough, the right hand side is positive, which proves (17).

It remains only to prove (14). Clearly by construction the following inequality is valid

$$M(x) \leq G(x) \leq H(x);$$

we deduce immediately the relations:

$$I(0,v_{k-1},0.5,p_k',M(x)) \leq I(-2,2,a,p_k',G(x)) \quad \text{for} \quad a < 0.5; \tag{19}$$

$$I(-2,2,1,p_k',G(x)) \leq I(-2,2,1,p_k',H(x)); \tag{20}$$

(14) is an immediate consequence of (18), (19), (20).

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